

Write on the board. Ask the class to find the total in 10 seconds without calculators.

Pair the numbers from opposite ends:

There are pairs in total: .

Summation

Sigma Notation

What is Sigma Notation?

Sigma Notation is a shorthand way of expressing a sum. It Uses the Greek letter *sigma*: Σ

The diagram shows the sigma notation $\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_n$. Callout boxes explain the parts:

- A box pointing to the upper n says: n is upper sur
- A box pointing to the lower $k=1$ says: k is called the index of summation
- A box pointing to a_k says: a_k is the formula used to generate each term of the sum

- The Gre
the gen

implify notation when

Any letter can be used to index a summation

$$\sum_{i=1}^n a_i = \sum_{j=1}^n a_j = \sum_{k=1}^n a_k$$

What is the value of $\sum_{j=1}^5 j^2$?

$$\begin{aligned}\sum_{j=1}^5 j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55.\end{aligned}$$

What is the value of $\sum_{k=4}^8 (-1)^k$?

$$\begin{aligned}\sum_{k=4}^8 (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1.\end{aligned}$$

Evaluate

$$\begin{aligned}\sum_{i=1}^4 (2i + 3) &= \overbrace{[2(1) + 3]}^{i=1} + \overbrace{[2(2) + 3]}^{i=2} + \overbrace{[2(3) + 3]}^{i=3} + \overbrace{[2(4) + 3]}^{i=4} \\ &= (2 + 3) + (4 + 3) + (6 + 3) + (8 + 3) \\ &= 5 + 7 + 9 + 11 \\ &= \mathbf{32}\end{aligned}$$

Evaluate

$$\begin{aligned}\sum_{j=3}^6 (2j + 3) &= \overbrace{[2(3) + 3]}^{j=3} + \overbrace{[2(4) + 3]}^{j=4} + \overbrace{[2(5) + 3]}^{j=5} + \overbrace{[2(6) + 3]}^{j=6} \\ &= (6 + 3) + (8 + 3) + (10 + 3) + (12 + 3) \\ &= 9 + 11 + 13 + 15 \\ &= \mathbf{48}\end{aligned}$$

Algebraic Properties & Closed-Form Formulas

Properties of Summations

Let c be a constant, and let a_n and b_n be sequences:

Linearity (Addition):

Constant Multiple:

Constant Sum:

Evaluate using $\sum$, $\prod$, and $\int$:

Expand terms directly before adding:

Split into two separate sums:

Evaluate using constant , , and :

Multiply each term by the constant first:

Factor the constant out of the sum:

Evaluate using constant and :

Add the constant repeatedly for iterations:

Multiply the constant directly by :

Essential Closed-Form Summation Rules

Sum of Integers:

Finite Geometric Sum:

Evaluating a Sum Symbolically

Evaluate without expanding the terms.

Solution:

Double Summations

Double summations iterate an inner loop completely for each step of an outer loop.

Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^4 \sum_{j=1}^3 ij.$$

$$\sum_{i=1}^4 \sum_{j=1}^3 ij = \sum_{i=1}^4 (i + 2i + 3i)$$

$$= \sum_{i=1}^4 6i$$

$$= 6 + 12 + 18 + 24 = 60.$$

Independent Bounds

Evaluate .

Solution:

Evaluate the inner sum with respect to :

Evaluate the outer sum with respect to :

Dependent bounds matching nested code loops

Evaluate inner sum:

Evaluate outer sum:

The loop executes n times, giving a time complexity of $O(n^2)$.

We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the $f(s)$, for all members s of S .

What is the value of $\sum_{s \in \{0,2,4\}} s$?

Evaluate

Expand the terms for each element in the set

Perform the addition

TABLE 2 Some Useful Summation Formulae.

<i>Sum</i>	<i>Closed Form</i>
$\sum_{k=0}^n ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, \ r \neq 1$
$\sum_{k=1}^n k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^n k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^n k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \ x < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \ x < 1$	$\frac{1}{(1-x)^2}$

Finite Geometric Series

Formula

Problem: Evaluate .

Solution: Identify parameters: , , .

Sum of First Integers (with Linearity)

Formula:

Problem: Find a closed-form expression for .

Solution: Apply linearity properties to split the sum, then apply the formula:

Sum of Squares

Formula:

Problem: Calculate the sum of squares from to .

Solution: Substitute :

Sum of Cubes

Formula:

Problem: Calculate .

Solution: Substitute :

Infinite Geometric Series

Formula:

Problem: Evaluate .

Solution: Identify . Since , the series converges:

Differentiated Infinite Geometric Series

Formula:

Problem: Evaluate .

Solution: Identify . Since , apply the closed-form identity:

Evaluate

To apply standard summation formulas, rewrite the sum as the total sum up to n minus the unneeded terms up to k :

Recall the sum of squares formula

Calculate each term

First sum (S_1):

Second sum (S_2):

Subtract the sums

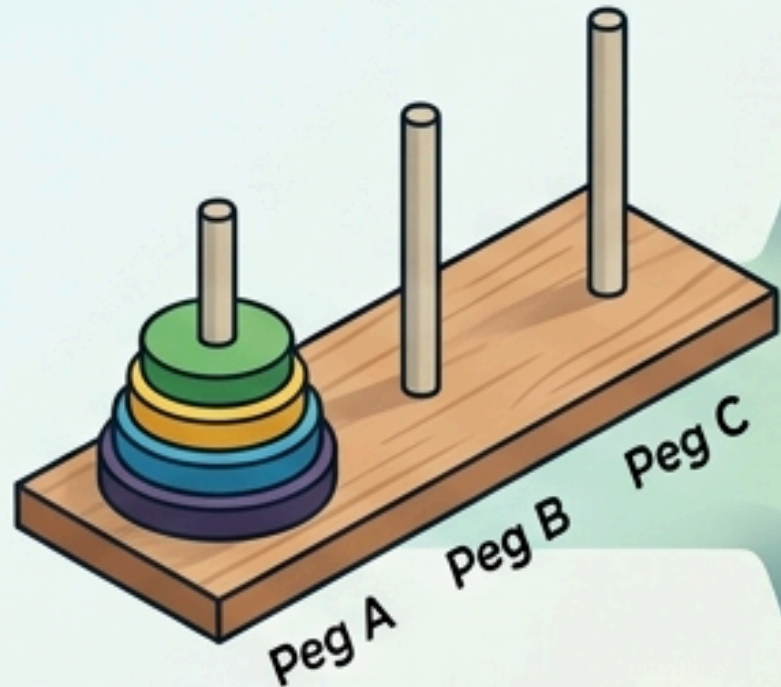
Recurrence Relations & Iteration

TOWER OF HANOI TO RECURRENCE: PHYSICAL MODEL & EXPERIMENTATION

Phase 2: The Physical Model — Tower of Hanoi to Recurrence

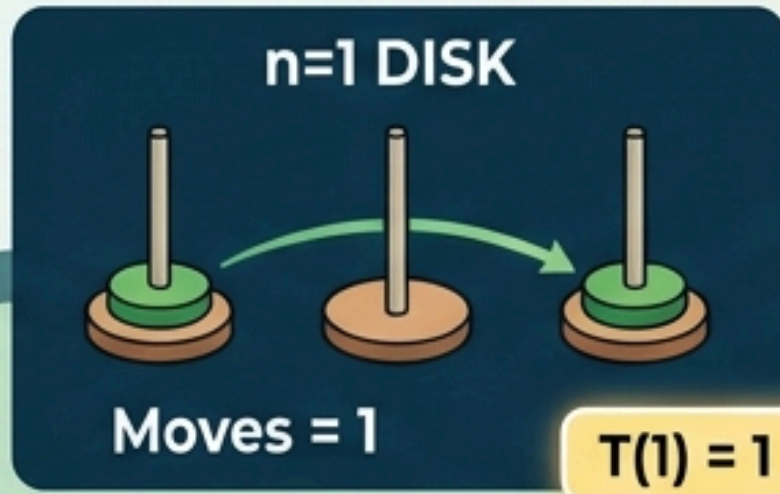
THE SETUP

Move all disks to another peg.
Rules: 1 disk at a time, no larger disk on a smaller disk.



Move all disks to another peg.
Rules: 1 disk at a time, no larger disk on a smaller disk.

TESTING SMALL VALUES OF n



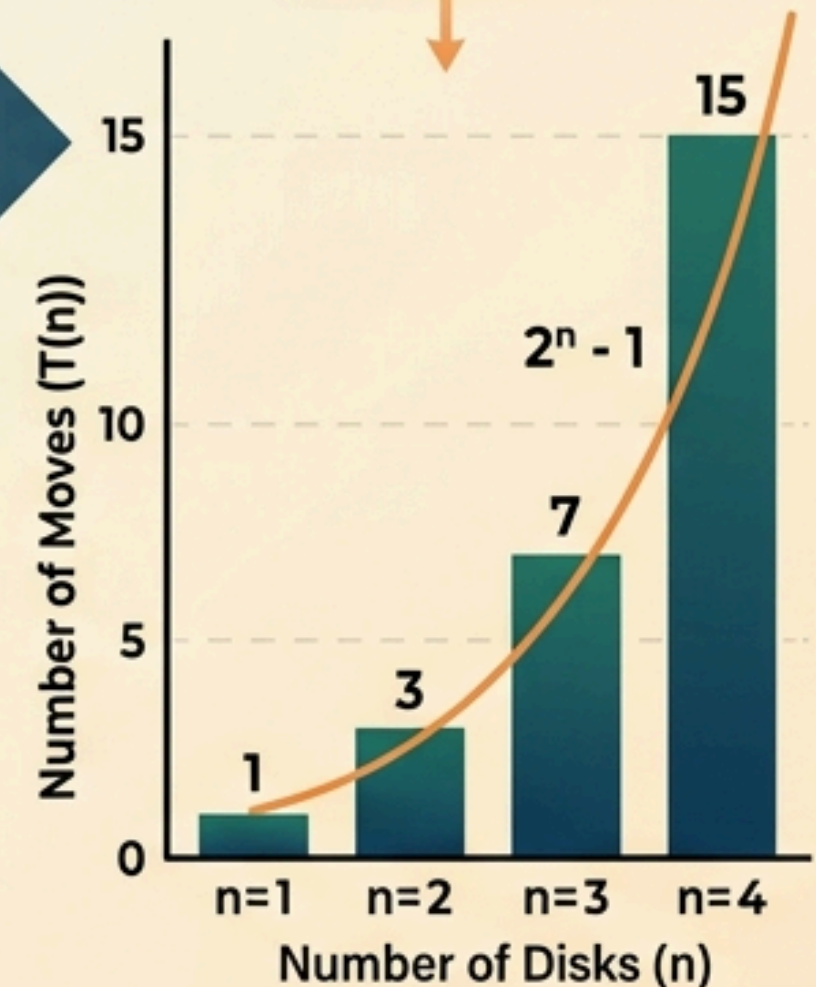
$$2 * T(1) + 1 = 3$$

$$2 * T(2) + 1 = 7$$

$$2 * T(3) + 1 = 15$$

TO RECURRENCE RELATION

$$T(n) = 2 * T(n-1) + 1$$
$$T(1) = 1$$



Definitions and Initial Conditions

A **recurrence relation** for a sequence is an equation that expresses the n -th term as a function of one or more preceding terms a_k for all integers $n \geq k$, where k is a nonnegative integer.

The **initial conditions** are the explicitly defined starting values that precede a_k . A recurrence relation alone defines infinitely many sequences; specifying its initial conditions uniquely selects a single sequence.

A **closed-form formula** (or explicit formula) calculates any term directly using n without requiring the computation of any preceding terms.

Solving Recurrence Relations by Iteration

Iteration derives a closed-form formula by repeatedly applying the recurrence relation.

Forward Iteration: Computes terms sequentially from initial conditions (n_0) until an algebraic pattern emerges.

Backward Iteration (Substitution Method): Replaces $T(n)$ with its expression in terms of $T(n-1)$, then $T(n-1)$ with its expression in terms of $T(n-2)$, stepping backward until reaching $T(n_0)$.

Forward Iteration (Linear/Additive Pattern)

Solve the recurrence relation for with initial condition .

Step 1: Compute successive terms

Step 2: Identify the pattern For any term , the common difference is added times to .

Step 3: State the explicit closed-form formula

Backward Iteration (Exponential/Geometric Pattern)

Solve the recurrence relation for with initial condition .

Step 1: Apply repeated backward substitution

Substitute

Substitute

Step 2: Generalize for steps

Step 3: Express in terms of Set :

Step 4: Substitute

Summation Pattern (Algorithm Step Counting)

Solve for with initial condition .

Step 1: Apply backward substitution

Step 2: Generalize down to

Step 3: Substitute and apply summation closed-form

The Fibonacci Sequence

The **Fibonacci sequence** is a second-order linear recurrence relation where each term is the sum of the previous two terms.

Recurrence Relation:

Initial Conditions:

Evaluating Terms Step-by-Step

Higher-Order Linear Recurrences: Second-Order Linear Recurrence with Coefficients

Given the recurrence for with initial conditions and , compute the first 6 terms of the sequence.

Computing terms sequentially:

Sequence values:

Solve the recurrence relation for with initial condition .

Step 1: Apply backward substitution

Substitute :

Substitute :

Step 2: Generalize for steps

Step 3: Express in terms of (Set)

Step 4: Apply and geometric series sum formula